The Two Bicliques Problem is in NP ∩ coNP

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Abstract

We show that the problem of deciding whether the vertex set of a graph can be covered with at most two bicliques is in NP∩coNP. We thus almost determine the computational complexity of a problem whose status has remained open for quite some time. Our result implies that a polynomial time algorithm for the problem is more likely than it being NP-complete unless P = NP.

keywords: Bicliques, Polynomial Time Algorithms, NP, coNP

1 Introdution

The problem of covering *the vertex set* of a graph with a minimum number of bicliques is one of the basic problems of graph theory with numerous applications of both theoretical and practical importance [19, 29, 31, 33, 34, 37]. Heydari, Morales, Shields Jr., and Sudborough show that the corresponding decision problem of determining whether a graph can be covered with at most k bicliques is NP-complete [21]. Indeed, Fleischner, Mujuni, Paulusma, and Szeider show that this decision problem remains NP-complete even when k is a fixed integer greater than two and not part of the input [17].

Interestingly, the complexity of deciding whether the vertex set of a graph can be covered with at most two bicliques has remained a challenging open problem. In particular, any theoretical evidence in favor of the problem either having an efficient algorithm or being NP-complete has remained elusive; see, for instance, [2, 10, 14, 17, 21]. In fact, Figueiredo classifies this problem, among a few others, as one of the important problems even in the P versus NP arena [14].

In this paper, we establish that this problem is in NP∩coNP. This effectively settles the problem in favor an efficient algorithm. For we learn from computational complexity

theory that such a problem is least likely to be NP-complete. For otherwise, the polynomial hierarchy is known to collapse to the first level [18, 36]. And problems that were seen to be in NP∩coNP have invariably been found subsequently to be in P as well [36].

Despite the fact that the problem allows efficient algorithms for several special classes of graphs [2, 11, 10, 17], our result still comes as a surprise for at least two reasons: (i) The closely related problem of deciding whether the vertex set of a connected graph can be covered with two P_4-free graphs is shown to be NP-complete by Hòang and Le [20]. (ii) Deciding whether a graph can be covered with two bicliques is essentially equivalent to deciding whether a connected graph has a disconnected vertex cut (see Lemma 2.5 or [17], for instance) but the closely related problem of deciding whether a connected graph has an independent vertex cut is known to be NP-complete [4, 6, 27]. [But a clique vertex cut is known to have a polynomial time algorithm [41].]

Note: Covering the vertex set of a graph with a minimum number of bicliques turns to be equivalent to partitioning the vertex set of the underlying graph into a minimum number of parts so that the induced subgraph on each part is covered by exactly one biclique. Therefore, by *partitioning a grpah into a minimum number of bicliques*, we essentially mean covering the vertex set of the graph with a minimum number of bicliques.

Notation: We denote by BPk the set of all graphs G such that G can be *partitioned* into at most k bicliques or, equivalently, such that the vertex set of G can be covered with at most k bicliques.

By $\overline{\mathrm{BP}k}$, we denote the set of all graphs G such that $G \notin \mathrm{BP}k$. Equivalently, $\overline{\mathrm{BP}k}$ is the set of all graphs G such that every partition of G into bicliques has more than k parts.

We use BP for denoting the set of all pairs (G, k) such that the graph G can be partitioned into at most k bicliques.

By convention, we will use BPk, $\overline{\mathrm{BP}k}$, and BP for denoting the membership problems associated with these sets.

Related Work: Bein, Bein, Meng, Morales, Shields Jr., and Sudborough show that it is NP-hard to find a c-approximation algorithm for BP for any constant c, apart from presenting a polynomial time exact algorithm for BPk restricted to bipartite graphs and restricted to certain other families of graphs [2].

The result of Fleischner, Mujuni, Paulusma, and Szeider that BPk is NP-complete for each fixed $k \geq 3$ also rules out a fixed parameter tractable algorithm for BP unless P = NP [17]. They moreover show that a certain natural bounded version of BP remains NP-complete and is W[2]-complete [12]. In contrast, they show the edge set version of biclique cover and biclique partition problems, which are known to be NP-complete [24, 32, 35] to be fixed parameter tractable. Their work includes a polynomial time algorithm for BP2 restricted to a family of graphs that includes bipartite graphs.

Recently, Dantas, Maffray, and Silva provide a list of several natural families of graphs such that there is a polynomial time algorithm for BP2 when restricted to graphs in each

of these families [10]. The list of families of graphs they consider includes K_4-free graphs, diamond-free graphs, planar graphs, bounded treewidth graphs, claw-free graphs, and (C_5, P_5)-free graphs.

Bicliques are one of the most sought-after structures of graphs, mainly due to their importance in applications, and has given rise to numerous computational problems involving bicliques from diverse branches of science; please consult the references.

2 Preliminaries

In this paper, we consider finite undirected simple graphs. We begin by formally defining a biclique as well as a star of a graph.

Definition 2.1 1. *A subgraph H of a graph G is said to be a* biclique *if H is isomorhic to either the complete graph K_1 or the complete bipartite graph $K_{m,n}$ for some $m, n \geq 1$.*

2. *A biclique H of a graph G is said to be a* star *if H is isomorphic to either the complete graph K_1 or the complete bipartite graph $K_{1,n}$ for some $n \geq 1$. The* center *of a star H is defined naturally.*

We now review the standard graph theory terminology and notation that we use.

Definition 2.2 1. *$\bar{G}$ denotes the complement of a graph G.*

2. *The empty graph on n vertices is denoted by nK_1: $nK_1 = \bar{K}_n$.*

3. *For a graph $G = (V, E)$ and $v \in V$, $N_G(v)$ denotes the set all vertices that are adjacent to v. [$N(v)$ does not include v.] We define $N_G[v] = N_G(v) \cup \{v\}$. We use $N(v)$ and $N[v]$ for these sets when G is understood.*

4. *For a graph $G = (V, E)$ and a set $A \subseteq V$, $G[A]$ denotes the induced subgraph of G on the vertices of A.*

5. *For a graph G and a vertex v of it, $G - v$ denotes the induced subgraph on $V(G) \setminus \{v\}$.*

6. *For a graph $G = (V, E)$ and a set $A \subseteq V(G)$, $G - A$ denotes the induced graph on $V(G) \setminus A$.*

7. *A vertex v of a connected graph G is said to be a* cut vertex *if $G - v$ is disconnected.*

8. *A set X of vertices of a connected graph G is said to be a* vertex cut *if $G - X$ is disconnected.*

We record a simple characterization of BP2 that is in the folklore. We state and prove it for compleness. Naturally, it turns to be a characterization for $\overline{\text{BP2}}$ as well. We begin with the following.

Lemma 2.3 *A graph $G \neq K_1$ is in BP1 if and only if $\bar{G}$ is disconnected.*

Proof: Let $G \in$ BP1. Then it possible that $G = K_1$; otherwise let $[A, B]$ be a partion of $V(G)$ such that each vertex of A is connected to every vertex of B. Then the complement graph $\bar{G}$ has no vertex of A connected to any vertex of B.

Conversely, if $G = K_1$ then it is a trivial biclique and belongs to BP1. Otherwise, assume that $\bar{G}$ is disconnected and set A to the set of vertices of a connected component of $\bar{G}$ and B to $V(G) \setminus A$. It follows that there is a biclique structure across A and B and so $G \in$ BP1. ■

Lemma 2.4 *A graph $G \neq 2K_1$ is in BP2$\setminus$ BP1 if and only if $\bar{G}$ is connected but has either a cut vertex or a disconnected vertex cut.*

Proof: Let G be a graph such that $G = 2K_1$ or $\bar{G}$ is connected but has a cut vertex or a disconnected vertex cut. Since $G = 2K_1 \in$ BP2 $\setminus$ BP1, we shall assume that $G \neq 2K_1$ and that $\bar{G}$ is connected. Then, clearly $G \notin$ BP1 by Lemma 2.3.

If $\bar{G}$ has a cut vertex, say v, then $\overline{G - v} = \bar{G} - v$ is disconnected and therefore, by Lemma 2.3, $G - v$ belongs to BP1. So, we conclude that $G \in$ BP2 $\setminus$ BP1.

If $\bar{G}$ has a disconnected vertex cut C, i.e., C is a vertex cut of $\bar{G}$ such that both $\bar{G}[C]$ and $\bar{G}[V(G) \setminus C]$ are disconnected, then both $G[C]$ and $G[V(G) \setminus C]$ are in BP1 by Lemma 2.3. So, we again conclude that $G \in$ BP2 $\setminus$ BP1.

Conversely, suppose that $G \in$ BP2$\setminus$BP1 and is not equal to $2K1$. Then $\bar{G}$ is necessarily connected; otherwise $G \in$ BP1 by Lemma 2.3.

If G has a two biclique partition with one of the parts as a single vertex, say v, then $G - v$ can be covered with one biclique which implies that $\bar{G} - v$ is disconnected, where we started with a $\bar{G}$ that is connected. Therefore v must be a cut vertex of $\bar{G}$.

If G allows a two biclique partition where neither of the bicliques is a single vertex, then $\bar{G}$ must be partitionable into two sets A and B such that both A and B have at least two elements each and $\bar{G}[A]$ and $\bar{G}[B]$ are disconnected. But $\bar{G} = \bar{G}[A \cup B]$ is connected. Therefore, it must be that A (as well as B) is a disconnected vertex cut of A. ■

Combining the preceding lemmas, we have the following.

Lemma 2.5 *A graph G that is not equal to K_1 or $2K_1$ is in BP2 if and only if one of the following is true: (a) $\bar{G}$ is disconnected; (b) $\bar{G}$ is connected but has a cut vertex; (c) $\bar{G}$ is connected but has a disconnected vertex cut.*

Consequently, we have the following lemma for graphs not in BP2.

Lemma 2.6 *A graph G on $n \geq 3$ vertices is in $\overline{BP2}$ if and only if $\bar{G}$ is connected, is free of cut vertices, and has all vertex cuts (if any) connected.*

The corollary below follows trivially from the lemma.

Corollary 2.7 *Let G be a graph in $\overline{BP2}$. Then the following are true for the complement graph $\bar{G}$.*

1. *The neighbours of any vertex of $\bar{G}$ induces a connected subgraph of $\bar{G}$ and this subgraph has at least two vertices.*

2. *From any vertex of $\bar{G}$, all other vertices are at most at a distance of two.*

3. *Any nonadjacent pair of vertices of $\bar{G}$ have a common neighbour in $\bar{G}$.*

We close the section with a definition that encapsulates an important notion that is central to our discussion.

Definition 2.8 *Let $\mathbf{F}$ be a family of graphs and let $G \in \mathbf{F}$. Let π be a permutation of a set $A \subseteq V(G)$ with $|A| = k$. Then π is said to be* safe for $\mathbf{F}$ *if each of $G_0, G_1, G_2, \ldots, G_k \in \mathbf{F}$, where G_i is the graph obtained from G by deleting all the vertices in a prefix of length i of π for each $0 \leq i \leq k$.*

3 Graphs of BP2 \ BP1

We show that from any graph G in BP2 \ BP1, by repeated deletion of zero or more vertices, we eventually and *inescapably* end up with a graph G' in BP2 \ BP1 that admits a partition into a star and a biclique, without ever leaving BP2 \ BP1 in the process. But we begin by proving the following Theorem.

Theorem 3.1 *Let G be a graph in BP2 \ BP1. Then we can decide whether G allows a star-biclique partition in polynomial time.*

Proof: Let G be a graph in BP2 \ BP1. Then for each vertex v of G, we simply check whether G admits a partition into a star biclique *centered at* v and another biclique. We do this as follows by fixing v for a particular vertex of G.

If G is disconnected, then there must be exactly two components. We simply check if at least one of the components is a star with v as the center; this can be done in polynomial time. So, we shall assume that G is connected.

If $G - v \in$ BP1, then v and $G - v$ provides a star-biclique partition of G. If $G - N[v] \in$ BP1, then $G[N[v]]$ and $G - N[v]$ provides a star-biclique partition of G.

If neither is the case, we decide in polynomial time whether there is a proper subset $S \neq \emptyset$ of $N_G(v)$ such that deleting $\{v\}$ and S from G results in a graph in BP1. For if there is such an S, then $G[\{v\} \cup S]$ and $G - v - S$ provides a star-biclique partition.

Since neither $G - v$ nor $G - N_G[v]$ is in BP1, both $G - v$ and $G - N_G[v]$ contain at least two vertices and the complement graphs $\bar{G} - v$ and $\bar{G} - N_G[v]$ are connected. Let $A = N_G(v)$ and let $B = V(G) \setminus N_G[v]$. Clearly, $A \cup B = V(G) \setminus \{v\}$.

Consider the complement graph $\bar{G} - v$. Let S be the set of all vertices u in A such that u is adjacent to some vertex in B in this complement graph. We note that this S can be

constructed in polynomial time. If $S = A$, [i.e., if each vertex of A is adjacent to a vertex in the connected graph $\bar{G} - N_G[v]$], then deleting no subset of A can disconnect $\bar{G} - v$; we shall therefore conclude that it is impossible to partition G into a star centered at v and a biclique.

If $S \neq A$, then S is a vertex cut for $\bar{G} - v$ and $\{v\} \cup S$ is a disconnected vertex cut for $\bar{G}$ with v as a component (No vertex in $S \subseteq A = N_G(v)$ is adjacent to v in $\bar{G}$.). In this case, we see that $G[\{v\} \cup S]$ and $G - v - S$ provide a star-biclique partition of G. ■

We have the following interesting result about graphs of BP2 \ BP1 that do not admit a star-biclique partition.

Lemma 3.2 *Let G be a graph in BP2 \ BP1 such that it does not admit any star-biclique partition. Then for any vertex v of G, $G - v$ is also a graph in BP2\BP1.*

Proof: Suppose that G does not allow any two biclique partition for which one of the bicliques is a star.

Then each biclique in every two biclique partition of G has on each side at least two vertices. So, deleting a vertex v from G does still retain a two biclique structure in $G - v$; and so $G - v \in$ BP2.

Since assuming that $G - v \in$ BP1 implies that G admits a star-biclique partition, namely v and $G - v$, we conclude that $G - v \in$ BP2 \ BP1. ■

The following theorem is a corollary of the above lemma.

Theorem 3.3 *For each graph G in BP2 \ BP1, there is an integer $l = l(G) \geq 0$ such that any permutation π of any subset of l vertices of G is safe for BP2 \ BP1. Moreover, none of the associated graphs $G_0, G_1, G_2, \ldots, G_{l-1}$ allows a star-biclique partition whereas the graph G_l does.*

4 Graphs of $\overline{\text{BP2}}$

The following theorem asserts that for any graph $G \in \overline{\text{BP2}}$, there is a careful order of deletion of vertices from G so that each of the successively resulting subgraphs is in $\overline{\text{BP2}}$ and the last graph H obtained is the smallest graph in $\overline{\text{BP2}}$, namely $3K_1 = \bar{K}_3$.

Theorem 4.1 *Let G be a graph in $\overline{BP2}$ on n vertices. Then G has a permutation π of $n-3$ vertices that is* safe *for $\overline{BP2}$.*

Proof: Let $G = (V, E)$ be a graph in $\overline{\text{BP2}}$ on n vertices. We will construct a permutation $\pi = \langle v_1, v_2, \ldots, v_{n-3} \rangle$ of $n - 3$ vertices of G that is *safe* for $\overline{\text{BP2}}$: deleting vertices in any prefix of π from G leaves behind a graph in $\overline{\text{BP2}}$.

Let A be a subset of V of largest cardinality such that the induced subgraph $G[A] \in$ BP2. In fact, the maximality of A implies that $G[A] \in$ BP2 \ BP1. Let $v \in V \setminus A$. Then

$G[A\cup\{v\}] \in \overline{\mathrm{BP2}}$. Clearly, deleting vertices in $V\setminus(A\cup\{v\})$ from G, in any order, can never result in a graph in BP2. We set π' equal to some ordering of vertices in $V\setminus(A\cup\{v\})$.

For every partition $[A_1, A_2]$ of A such that both $G[A_1]$ and $G[A_2]$ are in BP1, we have at least one vertex in A_1 that is not adjacent to v and at least one vertex in A_2 that is not adjacent to v. In fact, we have that $G[A_1\cup\{v\}] \notin \mathrm{BP1}$ and that $G[A_2\cup\{v\}] \notin \mathrm{BP1}$. For otherwise we will have that $G[A\cup\{v\}] \in \mathrm{BP2}$.

Let B be a subset of A of largest cardinality such that both $G[B]$ and $G[C]$, where $C = A\setminus B$, are in BP1. Then it follows, from the maximality of B that for each $c \in C$, there is at least one vertex $b \in B$ such that c is not adjacent to b. From what we noted in the preceding paragraph it also follows that v is not adjacent to some vertex in B and to some vertex in C.

We now delete all vertices in C that are adjacent to v in some order. It is clear that the sequence of successive graphs that are resulting are all in $\overline{\mathrm{BP2}}$. We continue deleting the other vertices of C except for one, say u, and note again that the successively resulting graphs are all in $\overline{\mathrm{BP2}}$. Let p'' denote the sequence of vertices deleted in the order of deletion. Let $H = G[B\cup\{u\}\cup\{v\}]$ denote the final graph obtained.

We note that vertices v and u are not adjacent in $H = G[B\cup\{u\}\cup\{v\}]$. Both v and u have nonadjacent vertices in B. Delete in some order all the vertices in B adjacent to v or u or both from H. When this is done, vertices v and u become isolated. We now continue deleting the other vertices of B except for one, say w, in some order. Let π''' be the sequence of vertices deleted. It is clear again that all the graphs obtained after each additional deletion are all in $\overline{\mathrm{BP2}}$.

We now set $\pi = \pi'\cdot\pi''\cdot\pi'''$ and see that π is a sequence of $n-3$ vertices of $G \in \overline{\mathrm{BP2}}$ on n vertices and that π is safe for $\overline{\mathrm{BP2}}$. ∎

5 Proving that BP2 $\in$ coNP

We establish that BP2 is in coNP by showing that $\overline{\mathrm{BP2}}$ is in NP. We provide a polynomial time verifier that takes in as input a graph G and a sequence π of vertices of G. The verifier accepts the pair if and only if $G \in \overline{\mathrm{BP2}}$ and π is safe for $\overline{\mathrm{BP2}}$ and is of length $n-3$, where $n = |V(G)|$. We know, from Theorem 4.1, that such a proof exists for all graphs in $\overline{\mathrm{BP2}}$.

Theorem 5.1 *There is a polynomial time algorithm that inputs a pair (G,π) of a graph G and a sequence π of vertices of G and outputs* ACCEPT *if and only if $G \in \overline{BP2}$ and π is a longest permutation of vertices of G that is safe for $\overline{BP2}$; it otherwise outputs* REJECT.

Proof: Consider the algorithm in Figure 1. We argue that this algorithm provides a valid polynomial time verifier for $\overline{\mathrm{BP2}}$. It is clear, from Theorem 3.1, that the algorithm can run in polynomial time. We will just prove its correctness.

Suppose that (G,π) is input to the algorithm.

If either $G \in \mathrm{BP1}$ or π is not obviously a longest safe sequence, the pair (G,π) is rightly rejected in Step 0.

Input: (G, π)
Output: ACCEPT / REJECT

0. If $G \in$ BP1 or π is *not* a permutation on $n-3$ vertices of the n vertex graph G, return REJECT. Else **repeat** Steps 1 to 3 below:

1. If G admits a star-biclique partition, return REJECT.

2. If $G = 3K_1$, return ACCEPT.

3. Remove the first vertex, v, from π and set $G = G - v$.

Figure 1: A Polynomial Time Verifier for $\overline{\text{BP2}}$

If $G \in \text{BP2} \setminus \text{BP1}$, then any repeated removal of zero or more vertices from G eventually necessarily results in a graph H that allows a star-biclique partition (Theorem 3.3) before giving rise to any graph that is probably not in BP2. Step 1 therefore ensures that no graph $G \in \text{BP2} \setminus \text{BP1}$ ever leads to the acceptance of the pair (G, π) with any false safe sequence π by detecting as and when a star-biclique structure arises from such a G; we know from Theorem 3.1 that this deduction can be carried out in polynomial time.

If $G \in \overline{BP2}$ but π is not safe for $\overline{BP2}$, then π has a prefix whose removal from G results in a graph H in BP2. If H does not admit a star-biclique partition, then continuing the removals further must (as argued in the preceding paragraph) eventually result in a graph that admits such a partition before possibly resulting in a graph that is not in BP2. Step 2 therefore also ensures that no wrong safe sequence π even with a $G \in \overline{BP2}$ leads to the acceptance of (G, π).

If G is a graph in $\overline{\text{BP2}}$ on n vertices and π is a permutation of $n-3$ vertices of π that is safe for $\overline{\text{BP2}}$ (such a sequence exists from Theorem 4.1), then π is necessarily a longest sequence that is safe for $\overline{\text{BP2}}$ and each subgraph of G obtained by deleting a prefix of π is in $\overline{\text{BP2}}$ and so none of them can clearly allow a star-biclique partition. Moreover, deleting all the vertices from such a π must necessarily result in $3K_1$; for this is the only graph on three vertices that is in $\overline{\text{BP2}}$. Therefore, such an input pair (G, π) is eventually rightly accepted, as can be easily verified, in Step 2 of the algorithm.

Steps 3 simply deletes the next vertex in π from G. The sequence π cannot be empty when the control enters Step 3 because it must have at least four vertices. For, if it has only three vertices, it must have either allowed a star-biclique partition already or been equal to $3K_1$ already; and the algorithm would have already stopped with an ACCEPT or a REJECT. ■

Conclusion

It remains an interesting open problem to see if the two biclique partition problem has a polynomial time algorithm. A negative answer to it, in particular, will resolve the famous P versus NP problem.

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